Contour gauges, canonical formalism and flux algebras.

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We would like to dedicate this paper to the memory of our dear friend and colleague Alyosha Anselm.

Abstract

A broad class of contour gauges is shown to be determined by admissible contractions of the geometrical region considered and a suitable equivalence class of curves is defined. In the special case of magnetostatics, the relevant electromagnetic potentials are directly related to the ponderomotive forces. Schwinger's method of extracting a gauge invariant factor from the fermion propagator could, it is argued, lead to incorrect results. Dirac brackets of both Maxwell and Yang-Mills theories are given for arbitrary admissible space-like paths. It is shown how to define a non-abelian flux and local charges which obey a local charge algebra. Fields associated with the charges differ from the electric fields of the theory by singular topological terms; to avoid this obstruction to the Gauss law it is necessary to exclude a single, gauge fixing curve from the region considered.

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1 Introduction

The Fock-Schwinger gauge [1, 2]

$$A \cdot x = 0 \tag{1.1}$$

has two remarkable features.
Firstly, the potentials can be expressed in terms of the fields [3]:

$$A_\mu^{F-S}(x) = \int_0^1 G_{\nu\mu}(xt)x^\nu t dt \tag{1.2}$$

where

$$G_{\mu\nu} = F_{\mu\nu} + ig[A_\mu, A_\nu] \tag{1.3}$$

Equation (1.2) is true both in Maxwell and Yang-Mills theories provided that the Bianchi identities are satisfied.
Secondly, the integral along *the straight line* segment $s(x,0)$ running from 0 to x.

$$\varphi_{s(x,0)}[A^{F-S}] = \int_{s(x,0)} A_\mu^{F-S}(y)dy^\mu \tag{1.4}$$

is constant i.e. independent of x

$$d\varphi_{s(x,0)} = \frac{\partial \varphi_s}{\partial x^\mu} dx^\mu = 0 \tag{1.5}$$

therefore

$$\varphi_{s(x,0)} = \varphi_{s(0,0)} \equiv 0 \tag{1.6}$$

Eqn.(1.5) can be also derived from (1.4) by direct calculation with the Bianchi identities taken into account and will be used by us in a more general context throughout this paper.
Equation (1.5) has an interesting physical interpretation. Take electrodynamics and the null loop $s(x,0)\bar{s}(x,0)$ i.e. the path followed along the straight segment $(0,x)$ and back. The flux $\Phi(s_x\overline{s_x}) = 0$ for any regular field $F_{\mu\nu}$. Next consider the deformed loop consisting of $s(x+\Delta x, 0)s(x, x+\Delta x)\bar{s}(x,0)$. The change of flux is gauge invariant and reads as:

$$\Delta\Phi = \varphi_{s(x+\Delta x,0)}[A] - \varphi_{s(x,0)}[A] + \varphi_{s(x,x+\Delta x)}[A] \tag{1.7}$$

Now $\varphi_{s(0,x)}$ does not depend on x for $A = A^{F-S}$, so

$$\Delta\Phi = \varphi_{s(x,x+\Delta x)}[A^{F-S}] \tag{1.8}$$

Taking $\Delta x \longrightarrow 0$ we get the infinitesmal change of the flux:

$$\Delta\Phi = -A_\mu^{F-S}(x)\Delta x^\mu \tag{1.9}$$

Therefore, $A_\mu^{F-S}(x)$ can be interpreted as a generalization of the magnetostatic ponderomotive field. Indeed, consider a closed circuit (current intensity I) which contains straight segment $s(\vec{x},0)$. Deformation of $s(\vec{x},0) \longrightarrow s(\vec{x}+\Delta\vec{x},0)s(\vec{x},\vec{x}+\Delta\vec{x})$ changes the magnetic interaction energy by $-I\Delta\Phi$ i.e. the ponderomotive force is $IA_i^{F-S}(\vec{x})$ in this three-dimensional example.

The Fock-Schwinger gauge is a special case of contour gauges [4], which in our paper will be specified by condition (1.5) with $s(x,0)$ replaced by more general family of curves $c(x,x_0)$. Let us notice that for any choice of $c(x,x_0)$ the corresponding potential f^c is again the ponderomotive field corresponding to the deformation $c(x,x_0)\bar{c}(x,x_0) \longrightarrow c(x+\Delta x,x_0)s(x,x+\Delta x)\bar{c}(x,x_0)$. Therefore it seems natural to name these path dependent gauges as ponderomotive ones. The ponderomotive interpretation nicely fits the fact that the contour gauges are a part of larger family of physical gauges; in electrodynamics a representant A^D of physical gauge we characterise by choice of the projection operator ${}^{\mathrm{D}}\hat{P}$

$${}^{\mathrm{D}}\hat{P}_\mu^\nu \cdot {}^{\mathrm{D}}\hat{P}_\nu^\rho = {}^{\mathrm{D}}\hat{P}_\mu^\rho \tag{1.10}$$

satisfying

$${}^{\mathrm{D}}\hat{P}_\mu^\nu \cdot (\partial_\nu\chi) = 0 \tag{1.11}$$

for the arbitrary scalar function $\chi(x)$.
Then

$$\mathcal{A}_\mu^{(D)} \equiv {}^{\mathrm{D}}\hat{P}_\mu^\nu \mathcal{A}_\nu \tag{1.12}$$

In particular

$$\mathcal{A}_\mu^{(D)} \equiv {}^{\mathrm{D}}\hat{P}_\mu^\nu \mathcal{A}_\nu^{(D)} \tag{1.13}$$

It follows from eqns.(1.10-1.12) that $\mathcal{A}^{(D)}$ is gauge invariant quantity with respect to transformation of the gauge field $A_\mu(x)$, but, on the other hand the eqn. (1.13) is a gauge condition for the choice of $\mathcal{A}^{(D)}$. A vast choice of different gauge invariant formalisms [5] has evidently it's source in the freedom of defining projection operators satisfying eqn.(1.11).

In this paper we shall present results concerning contour gauges, characterized by a broad class of curves $c(x, x_0)$; the corresponding projection operator $P^{c(x,x_0)}$ acts on the 4-potentials as follows

$$[\hat{P}^c \cdot \mathcal{A}]_\mu(x) = \mathcal{A}_\mu(x) - \frac{\partial}{\partial x^\mu} \int_{Y \in c(x,x_0)} dY^\nu \mathcal{A}_\nu(Y) \tag{1.14}$$

and as a result of eqn.(1.13) the chosen potential $f^{c(x,x_0)}$ has to satisfy eqn.(1.5) (with $s(x, 0)$ replaced by $c(x, x_0)$). In chapter 2 we show that for quite general class of curves (including non-smooth ones) our gauge condition determines $f^c[F]$ in the case of electrodynamics. In the same chapter a connection between possible contractions of the space-time region considered and the freedom of choice of our ponderomotive potentials is exhibited and a suitable equivalence class of curves is defined. A subclass of curves - suitable for Yang-Mills theory - is specified and applied in Ch.3. Some gauges which have been considered in this theory, such as Fock-Schwinger [3], superaxial [6], temporal with space-like Fock-Schwinger [7] are shown to belong to our class of gauges.

We generalize - in Ch.5 - Schwinger's method [2] of extracting a field dependent factor from the fermion propagator for a Dirac particle interacting with a given electromagnetic field. This enables us to provide an expansion for the fermion propagator in terms of the electromagnetic fields themselves. The consistency of the ponderomotive gauge constraints with the canonical formalism is exhibited in the Maxwell theory in Ch.6. The Dirac brackets can be obtained in a straightforward way due to the fact that our gauge constraint at fixed time is canonically conjugate to the Gauss law constraint.

The application of canonical formalism to Yang-Mills theories is discussed in chapters 7 - 9. The Dirac brackets are given in Ch.7. Flux operator algebras and local charge algebras with structure constants of underlying Yang-Mills theory are derived in chapters 8, 9. Fields associated with these charges differ from the electric fields of the theory by singular topological terms; to avoid this obstruction to the Gauss law it is necessary to exclude a single gauge fixing curve from the region considered in the theory.

2 Ponderomotive gauges - electrodynamics

Let us consider an arbitrary electromagnetic field

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1}$$

which is continuously once differentiable at any $x \in V_4$, V_4 being some open, simply connected space-time region. Next, let us associate with every pair (x_1, x_2) a unique continuous path running from $x_2 \in V_4$ to $x_1 \in V_4$ and contained in V_4:

$$(x_1, x_2) \longrightarrow c(x_1, x_2) \subset V_4 \tag{2.2}$$

The parametric representation of c is taken to be:

$$x \in c(x_1, x_2) \Longleftrightarrow x^\mu = c^\mu(x_1, x_2, \tau) \in V_4 \tag{2.3}$$

for some $\tau \in [0, 1]$ with

$$\begin{aligned} x_2^\mu &= c^\mu(x_1, x_2, 0) \\ x_1^\mu &= c^\mu(x_1, x_2, 1) \end{aligned} \tag{2.4}$$

Our results will not depend on the equivalent reparametrizations:

$$\tau \longrightarrow t(\tau) \tag{2.5}$$

with

$$\frac{\partial t}{\partial \tau} > 0, \tag{2.6}$$

In what follows we shall fix $x_2 = x_0$ and demand that $c^\mu(x, x_0, \tau)$ satisfy the following regularity conditions:

i) c^μ is $C^{(2)}$ - regular as a function of x, τ everywhere in $V_4 \times [0, 1]$ with the possible exception of $\tau = \tau_1, \ldots \tau_n$; $\tau_i(x)$ being $C^{(2)}$-regular.

ii) At $\tau \longrightarrow \tau_i$, at fixed x, the left - and right - side finite limits exist for $n \leq 2$ derivatives. These limits coincide for:

$$c^\mu(x, x_0, \tau)|_{\tau \to \tau_i(x)} = c^\mu(x, x_0, \tau_i(x)) \tag{2.7}$$

$$\frac{\partial c^\mu(x, x_0, \tau)}{\partial x^\rho}|_{\tau \to \tau_i} + \frac{\partial c^\mu}{\partial \tau}|_{\tau \to \tau_i} \frac{\partial \tau_i(x)}{\partial x_\rho} = \frac{\partial c^\mu(x, x_0, \tau_i(x))}{\partial x^\rho} \tag{2.8}$$

iii) In the limit $\tau \longrightarrow 0, 1$

$$c^\mu(x, x_0, \tau) \xrightarrow{\tau \to 1} x^\mu + O(1-\tau) g_1^\mu(x) \tag{2.9}$$

$$c^\mu(x, x_0, \tau) \xrightarrow{\tau \to 0} x_0^\mu + O(\tau) g_2^\mu(x) \tag{2.10}$$

The somewhat general condition ii) is of importance; it admits functions with discontinuously changing direction.

We shall now prove the following theorem:

Theorem 1 *Given the electromagnetic field $F_{\mu\nu}$, the condition*

$$d\varphi_{c(x,x_0)} = 0 \tag{2.11}$$

with

$$\varphi_{c(x,x_0)} = \int_{c(x,x_0)} A_\mu(y) dy^\mu \tag{2.12}$$

uniquely determines the electromagnetic potential A as

$$A_\rho = f_\rho^c(x) \equiv \int_{c(x,x_0)} F_{\nu\mu}(y) \frac{\partial y^\mu}{\partial x^\rho} dy^\nu \tag{2.13}$$

<u>*Proof*</u>
We can write

$$\varphi_{c(x,x_0)} = \sum_{i=0}^{n} I_i \tag{2.14}$$

with

$$I_i = \int_{\tau_i}^{\tau i+1} A_\mu(c) \frac{\partial c^\mu}{\partial \tau} d\tau \tag{2.15}$$

and

$$\begin{aligned} \tau_0 &= 0 \\ \tau_{n+1} &= 1 \end{aligned} \tag{2.16}$$

Next,

$$\frac{\partial I_i}{\partial x^\rho} = A_\mu(c) \frac{\partial c^\mu}{\partial \tau}|_{\tau=\tau_{i+1}(x)} \cdot \frac{\partial \tau_{i+1}(x)}{\partial x^\rho} - A_\mu(c) \frac{\partial c^\mu}{\partial \tau}|_{\tau=\tau_i(x)} \cdot \frac{\partial \tau_i(x)}{\partial x^\rho} + D \tag{2.17}$$

where

$$
\begin{aligned}
D &= \int_{\tau_i}^{\tau_{i+1}} [A_{\mu,\alpha} \cdot \frac{\partial c^\alpha}{\partial x^\rho} \cdot \frac{\partial c^\mu}{\partial \tau} + A_\mu \frac{\partial^2 c^\mu}{\partial x^\rho \partial \tau}] d\tau = \\
&= A_\mu(c) \frac{\partial c^\mu}{\partial x^\rho} |_{\tau=\tau_i}^{\tau=\tau_{i+1}} - \int_{\tau_i}^{\tau_{i+1}} F_{\nu\mu}(c) \frac{\partial c^\mu}{\partial x^\rho} \cdot \frac{\partial c^\nu}{\partial \tau} d\tau \qquad (2.18)
\end{aligned}
$$

Hence

$$
\frac{\partial I_i}{\partial x_\rho} = A_\mu(c) \frac{\partial c^\mu}{\partial x^\rho} |_{c=c(x,x_0),\tau_i(x))}^{c=c(x,x_0,\tau_{i+1}(x))} - \int_{\tau_i}^{\tau_{i+1}} F_{\nu\mu} \frac{\partial c^\mu}{\partial x^\rho} \cdot \frac{\partial c^\nu}{\partial \tau} d\tau \qquad (2.19)
$$

Therefore,

$$
\frac{\partial \varphi_{c(x,x_0)}}{\partial x^\rho} = A_\mu(c) \cdot \frac{\partial c^\mu}{\partial x^\rho} |_{c=c(x,x_0,0)}^{c=c(x,x_0,1)} - \int_0^1 F_{\nu\mu} \frac{\partial c^\mu}{\partial x^\rho} \frac{\partial c^\nu}{\partial \tau} d\tau \qquad (2.20)
$$

Using eqn.(2.4), (2.9) and (2.10) we get

$$
d\varphi_{c(x,x_0)}[A] = \left[A_\rho(x) - \int_0^1 F_{\nu\mu}(c) \frac{\partial c^\mu}{\partial x^\rho} \frac{\partial c^\nu}{\partial \tau} \right] dx^\rho \qquad (2.21)
$$

and from $d\varphi = 0$ eqn.(2.13) follows. Moreover, it can be directly checked that

$$
f^c_{\mu,\nu}(x) - f^c_{\nu,\mu}(x) = F_{\nu\mu}(x) \qquad (2.22)
$$

for f^c given by eqn.(2.13) provided that $F_{\mu\nu}$ is antisymmetric and satisfies the Bianchi identities and that c has to satisfy the regularity conditions i), ii), iii).

It seems rewarding that our gauge condition (2.11) leads to the expression (2.13) for A_μ, identical with a known illustration of the Poincare Lemma [8, 9]. In this context it is worth noticing that the converse of Theorem 1 is true and follows from eqn.(2.21) i.e. if we choose A in eq.(2.21) as f^c given by eqn.(2.13), then $d\varphi_c[f^c] = 0$ i.e. eqn.(2.11) is satisfied.
Let us remark that we tacitly demand that V_4 be contractible [9] to the point $x_0 \in V_4$ with the admissible deformations (homotopy) ζ defined through the conditions i), ii), iii) (of course $C^{(\infty)}$ - class homotopy satisfies our conditions, too). Therefore, both a set D of points x_2 to which V_4 is ζ - contractible and sets of admissible curve families $c^\mu(x, x_2 = const, \tau)$ for each $x_2 \in D$ are determined by V_4 itself. Let us label the relevant collection of all such

$c(x, x_2 \in D)$ by $\zeta(D)$.
To any $c \in \zeta$ there corresponds a unique electromagnetic potential $A = f^c$ satisfying eqn.(2.11). Let us check when different curves may correspond to the same f. Take two different curve families $c_1(x, x_{01})$, $c_2(x, x_{02})$ and assume

$$f^{c_1} = f^{c_2} \tag{2.23}$$

$$i.e.\ \ d\varphi_{c_1}[f^{c_1}] = d\varphi_{c_2}[f^{c_1}] = 0 \tag{2.24}$$

First, notice that

$$d\varphi_c[A] = 0 \Longleftrightarrow \varphi_{c(x,x_0)} = \varphi_{c(x_0,x_0)} = c[F] \tag{2.25}$$

where

$$c[F] = \oint_{c(x_0,x_0)} Adc \tag{2.26}$$

where $c[F]$ does not depend on x and $c[F] \neq 0$ unless $c(x, x_0)$ is contractible to a tree [10] i.e. to a closed path (e.g. a point) with "null area".
Next, from (2.24) and (2.25) it follows that

$$\int_{c_1(x,x_{01})\bar{c}_2(x,x_{02})} f^{c_1}(y)dy = c_1[F] - c_2[F] \tag{2.27}$$

for any x and therefore,

$$\oint_L f^{c_1}(y)dy = 0 \tag{2.28}$$

for any x, $x' \in V_4$;

$$L = c_1(x, x_{01})\bar{c}_2(x, x_{02})c_2(x', x_{02})\bar{c}_1(x', x_{01}) \tag{2.29}$$

As eqn.(2.28) has to be true for any chosen electromagnetic, $C^{(1)}$- regular field F, L has to be a tree in order to satisfy (2.28). Therefore we should consider a set of classes $\zeta(D)/T$ with equivalence relation T:

$$c_1(x, x_{01}) = c_2(x, x_{02}) \tag{2.30}$$

if, for any x, $x' \in V_4$ the closed path $L(c_1\bar{c}_2c_2^{,}\bar{c}_1^{,})$ (eqn. 2.29) is a tree [10].

(It can be checked that relation (2.30) is an equivalence relation) Choosing a single representative for each class c/T we get a one to one correspondence $c \longleftrightarrow f^c$. The permutation of a set $\zeta(D)/T$

$$P : c \longrightarrow P(c) \tag{2.31}$$

implies

$$f^c \longrightarrow Pf^c = f_{P(c)} \tag{2.32}$$

which, after the use of eqn.(2.13) and (2.21) (inserting $A = f^c$)reads

$$f^c(x)dx \longrightarrow f^{P(c)}(x)dx = f^c(x)dx - d\varphi_{P(c)}[f^c] \tag{2.33}$$

From eqns.(2.13),(2.21), we easily find that

$$d\varphi_{c_1}[f^{c_2}] + d\varphi_{c_2}[f^{c_1}] = 0 \tag{2.34}$$

so that (2.33) becomes

$$f^c(x)dx = f^{P(c)}(x)dx - d\varphi_c[f^{P(c)}] \tag{2.35}$$

Finally, comparing (2.33) with a familiar form

$$f^{P(c)}dx = f^c dx - \frac{i}{e}\ \omega^{-1}_{c\to P(c)} d\omega_{c\to P(c)} \tag{2.36}$$

we get

$$d\varphi_{P(c)}[f_c] = \frac{i}{e} d\ln\omega_{c\to P(c)} \tag{2.37}$$

i.e.

$$\omega_{c\to P(c)} = \omega_0 \exp(-ie\varphi_{P(c)}[f_c]) \tag{2.38}$$

Let us add, that the transition $A \longrightarrow f^c$ can be obtained similarly for an arbitrary potential A:

$$f^c = A - \frac{i}{e}\omega^{-1}(A \to f^c)\partial\omega(A \longrightarrow f^c) \tag{2.39}$$

with

$$\omega(A \longrightarrow f^c) = \omega_0 \exp(-ie\varphi_c[A]) \tag{2.40}$$

so that any A can be gauge transformed to f^c.

3 Yang-Mills theory

The generalization of the above results to Yang-Mills fields is relatively straightforward except for one complication linked to the Bianchi identity. In the Yang-Mills theories the field $G_{\mu\nu}$ is defined as

$$G_{\mu\nu} = F_{\mu\nu} + ig[A_\mu, A_\nu] \tag{3.1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.2}$$

and

$$A_\mu = A^a_\mu T_a \tag{3.3}$$

From (3.1), (3.2) the Bianchi identities follow:

$$D_\mu G_{\nu\rho} + D_\nu G_{\rho\mu} + D_\rho G_{\mu\nu} = 0 \tag{3.4}$$

where

$$D_\mu = \partial_\mu + ig[A_\mu, \cdot] \tag{3.5}$$

The condition (2.11) applied to the matrix A_μ:

$$d\varphi_{c(x,x_0)}[A] = 0 \tag{3.6}$$

for a curve with

$$c^\mu(x^0, x^0) = x^{0^\mu} \tag{3.7}$$

leads - due to eqns. (2.25),(2.26) - to

$$\varphi_{c(x,x_0)} = 0 \tag{3.8}$$

Therefore, trivially

$$\partial_\mu \varphi = D_\mu \varphi = 0 \tag{3.9}$$

Theorem 1 can be applied to A with a given F and, using (2.13) with (3.1) we get

$$f^c_\rho = \int_c \Big(G_{\nu\mu}(c) - ig[f^c_\nu, f^c_\mu] \Big) \frac{\partial c^\mu}{\partial x^\rho} \frac{\partial c^\nu}{\partial \tau} d\tau \tag{3.10}$$

In order to simplify (3.10) let us limit ourselves to a class of curve families satisfying as explained below a self-contractibility condition:

For any $y \in c(x, x_0)$,

$$c(x, x_0) \cap c(y, x_0) = c(y, x_0) \tag{3.11}$$

This condition means that the curves defined by:

$$y \in \tilde{c}(c(x, x_0, t), x_0) \Longleftrightarrow y^\mu = c^\mu(x, x_0, \tau_1) \tag{3.12}$$

for some $0 \leq \tau_1 \leq t$
and

$$y \in c(c(x, x_0, t), x_0) \Longleftrightarrow y^\mu = c^\mu(c(x, x_0, t), x_0, \tau) \tag{3.13}$$

for some $0 \leq \tau \leq 1$
are the same for any $0 \leq t \leq 1$. Therefore, there must exist a change of parametrization

$$\begin{gathered} \tau \longrightarrow h(\tau, x, t), \quad \frac{\partial h}{\partial \tau} > 0 \\ h(1, x, t) = t \\ h(0, x, t) = 0 \end{gathered} \tag{3.14}$$

leading to the following relation

$$c^\mu(c(x, x_0, t), x_0, \tau) = c^\mu(x, x_0, h(\tau, x, t)) \tag{3.15}$$

We are now in a position to prove the following theorem:

Theorem 2 *For any self-contractible family of curves, c, eqn.(3.1) together with the condition (3.6) are equivalent to:*

$$A_\rho = f_\rho^c(x) \equiv \int_{c(x,x_0)} G_{\nu\mu}(y) \frac{\partial y^\mu}{\partial x^\rho} dy^\nu \tag{3.16}$$

provided that $f^c[G]$, G satisfy the Bianchi identities (3.4).

Proof
Firstly let us prove that (3.16) follows from (3.1), (3.6). This will be achieved by the proof of the following lemma.
Lemma

For any self-contractible curve family c, and any antisymmetric $T_{\mu\nu}$, a vector field $f^c_\rho(x)$ defined as

$$f^c_\rho(x) = \int_{c(x,x_0)} T_{\nu\mu}(y) \frac{\partial y^\mu}{\partial x^\rho} dy^\nu \tag{3.17}$$

is perpendicular to the curves of this family:

$$\frac{\partial c^\rho(x, x_0, t)}{\partial t} f^c_\rho(c^\mu(x, x_0, t)) = 0 \qquad 0 \leq t \leq 1 \tag{3.18}$$

Proof of lemma
Let

$$R = \frac{\partial c^\rho(x, x_0, t)}{\partial t} f^c_\rho(c^\mu(x, x_0, t)) \tag{3.19}$$

Using (3.17) we get

$$R = \frac{\partial c^\rho(x, x_0, t)}{\partial t} \int T_{\nu\mu}(c(c(x, x_0, t), x_0, \tau)) \frac{\partial c^\mu(c(x, x_0, t), x_0, \tau)}{\partial c^\rho(x, x_0, t)} \frac{\partial c^\nu(c, x_0, \tau)}{\partial \tau} d\tau \tag{3.20}$$

Hence

$$R = \int T_{\nu\mu}(c(c, x_0, \tau)) \frac{\partial c^\mu(c(x, x_0, t), x_0, \tau)}{\partial t} \frac{\partial c^\nu(c, x_0, \tau)}{\partial \tau} d\tau \tag{3.21}$$

and using (3.15) we get

$$R = \int T_{\nu\mu}(c(x, x_0, h(\tau, x, t))) \frac{\partial c^\mu(x, x_0, h)}{\partial t} \frac{\partial c^\nu(x, x_0, h)}{\partial \tau} d\tau = = \int T_{\nu\mu} \frac{\partial c^\mu}{\partial h} \frac{\partial c^\nu}{\partial h} \frac{\partial h}{\partial t} \frac{\partial h}{\partial \tau} d\tau \tag{3.22}$$

i.e.

$$R = 0 \tag{3.23}$$

due to the antisymmetry of $T_{\mu\nu}$ in (3.22).
Inserting eqn.(3.18) into (3.10) we get

$$f^c_\rho = \int G_{\nu\mu}(c) \frac{\partial c^\mu}{\partial x^\rho} \frac{\partial c^\nu}{\partial \tau} d\tau \tag{3.24}$$

for any self-contractible family of curves c i.e. we get eqn.(3.16) of theorem 2. Eqn.(3.4) trivially follows from (3.1) for any A. The second part of the proof of Theorem 2 is given in Appendix A; we show there, that $f^c[G]$ satisfies eqns.(3.1), (3.6) if (3.4) is fulfilled.

For $c \longrightarrow c'$ (c, c' self-contractible)

$$f^{c'} = \omega^{-1}_{c\to c'} f^c \omega_{c\to c'} - \frac{i}{g}\omega^{-1}_{c\to c'}\partial\omega_{c\to c'} \tag{3.25}$$

with

$$\omega_{c\to c'} = P\exp\left(-ig\int_0^1 f^c(c')\frac{\partial c'}{\partial\tau}d\tau\right)\omega_0 \tag{3.26}$$

Proof: Take any $x = c'(x, x_0, t)$.then, from (3.18):

$$0 = \omega^{-1}_{c\to c'} f^c_\rho(c'(x, x_0, t))\omega_{c\to c'}\frac{\partial c^{\rho'}}{\partial t} - \frac{i}{g}\omega^{-1}_{c\to c'}\frac{\partial c'}{\partial t}\frac{\partial}{\partial c'}\omega_c \tag{3.27}$$

i.e.

$$f^c_\rho(c'(x, x_0, t))\frac{\partial c'^\rho}{\partial t}\omega_{c\to c'} = \frac{i}{g}\frac{\partial}{\partial t}\omega_{c\to c'} \tag{3.28}$$

This is solvable, and for $c'(x, x_0, 1) \equiv x$ gives (3.26). Moreover (3.26) remains true for any $A \longrightarrow f^{c'}$ transition, showing that any A can be gauge transformed into f^c.

4 Relation to other Yang-Mills gauges

We show now that those gauge conditions utilized [3, 6, 7] in Yang-Mills theory which lead to the form $A = A[G]$; are special case of our ponderomotive gauges coresponding to special cases of the curves c. If we take $c(0, x^\mu) = s(0, x^\mu)$, evidently satisfying condition (3.11) we get from (3.18)

$$A_\mu x^\mu = 0 \tag{4.1}$$

i.e. the Fock-Schwinger gauge [3].
For $c(0, x^\mu) = s(0, [0, \vec{x}])s([0, \vec{x}], [x^0, \vec{x}]$, where $[a, \vec{b}]$ denote a space-time point, we get temporal and space-like Fock-Schwinger [7] gauges:

$$\begin{aligned}\vec{A}(0, \vec{x}) \cdot \vec{x} &= 0 \\ A^0(x_0, \vec{x}) &= 0\end{aligned} \tag{4.2}$$

Finally, choosing

$$c(0,x^\mu)=s(0,[x^0,\vec{0}])s([x^0,\vec{0}],[x^0,x^1,0,0])s([x^0,x^1,0,0],[x^0,x^1,x^2,0])$$
$$s([x^0,x^1,x^2,0],[x^0,x^1,x^2,x^3]) \quad (4.3)$$

we get the superaxial gauge (see e.g.[6]):

$$\begin{aligned} A_0(x^0,\vec{0})=0 \qquad A_2(x^0,x^1,x^2,0)=0 \\ A_1(x^0,x^1,0,0)=0 \qquad A_3(x^\mu)=0 \end{aligned} \quad (4.4)$$

5 Path dependent Green's functions

Now we consider a Dirac particle interacting with a given electromagnetic field. Use of our ponderomotive gauges enables us to generalize Schwinger's method of extracting a gauge invariant factor from the fermion propagator and leads to a perturbative expansion for the propagator in terms of the electromagnetic field itself. We show that Schwingers approach to this problem could lead to incorrect results.
Starting with the Dirac equation

$$(\partial\!\!\!/ + ieA\!\!\!/ + im)\Psi(A\mid x)=0 \quad (5.1)$$

we substitute, for some family $c(x,x_0)$:

$$\Psi(f^c\mid x)=\omega(A\longrightarrow f^c)\Psi(A\mid x) \quad (5.2)$$

where (compare eqn.(2.40); we put $\omega_0=1$):

$$\omega(A\longrightarrow f^c)=\exp(-ie\varphi_c[A]) \quad (5.3)$$

Then, using (2.39) we get

$$(\partial\!\!\!/ + ief\!\!\!/^{\,c} + im)\Psi(f\!\!\!/^{\,c}\mid x)=0 \quad (5.4)$$

Similarly, the Green function

$$G(A\mid x_1,x_2)=-i<0\mid T(\Psi(A\mid x_1)\bar{\Psi}(A\mid x_2))\mid 0> \quad (5.5)$$

satisfying

$$(\partial\!\!\!/_1 + ieA\!\!\!/ + im)G(A\mid x_1,x_2)=-i\delta(x_1-x_2) \quad (5.6)$$

$$G(A \mid x_1, x_2)(-\overleftarrow{\not\partial}_2 + ie\not{A}(x_2) + im) = -i\delta(x_2 - x_1) \tag{5.7}$$

can be replaced by

$$G(f^c \mid x_1, x_2) = \omega(A \longrightarrow f^{c(x_1,x_0)})G^A\omega^{-1}(A \longrightarrow f^{c(x_2,x_0)}) \tag{5.8}$$

and eqns.(5.6), (5.7) are replaced by

$$(\not\partial_1 + ie\not{f}^c(x_1) + im)G(f^c \mid x_1, x_2) = -i\delta(x_1 - x_2) \tag{5.9}$$

$$G(f^c \mid x_1, x_2)(-\overleftarrow{\not\partial}_2 + ie\not{f}^c(x_2) + im) = -i\delta(x_1 - x_2) \tag{5.10}$$

They have the same iterative solution as (5.6), (5.7):

$$G(f^c \mid x_1, x_2) = S + eS\not{f}_cS + e^2S\not{f}_cS\not{f}_cS + \ldots \tag{5.11}$$

where

$$S(x, y) = \int \frac{d^4p}{(2\pi)^4}\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}e^{-ip(x-y)} \tag{5.12}$$

The above suggests that an approach used by Schwinger [2] may not be generally applicable. Schwinger uses a different transformation from ours (5.8) namely:

$$\widetilde{G}(x_1, x_2) = \omega(A \longrightarrow f^{c(x_1,x_2)})G(A \mid x_1, x_2) \tag{5.13}$$

Although the definitions (5.8) and (5.13) differ only by a (known) phase factor, the difference could become important in calculations involving derivatives of the Green's function. While in eqns.(5.9), (5.10) involving $G(f_c \mid x_1, x_2)$ only one function

$$f^{c,L}_{(x,x_0)} \equiv f^c(x, x_0) \tag{5.14}$$

occurs in both equations, the equations for $\widetilde{G}$ will involve two functions, $f^{c,L}$ and $f^{c,R}$:

$$(\not\partial_1 + ie\not{f}^{c,L}(x_1, x_2) + im)\widetilde{G}(x_1, x_2) = -i\delta(x_1 - x_2) \tag{5.15}$$

$$\widetilde{G}(x_1, x_2)(-\overleftarrow{\not\partial}_2 + ie\not{f}^{c,R}(x_1, x_2) + im) = -i\delta(x_1 - x_2) \tag{5.16}$$

where

$$f^{c,L}_{\rho}(x_1,x_2)=\int_{c(x_1,x_2)}F_{\nu\mu}(x)\frac{\partial x^{\mu}}{\partial x^{\rho}_1}dx^{\nu} \tag{5.17}$$

$$f^{c,R}_{\rho}(x_1,x_2)=\int_{c(x_1,x_2)}F_{\nu\mu}(x)\frac{\partial x^{\nu}}{\partial x^{\rho}_2}dx^{\mu} \tag{5.18}$$

Both these functions are potentials corresponding to the same $F_{\mu\nu}$ i.e.

$$\frac{\partial}{\partial x^{\sigma}_1}f^{c,L}_{\rho}(x_1,x_2)-\frac{\partial}{\partial x^{\rho}_2}f^{c,L}_{\sigma}(x_1,x_2)=F_{\sigma\rho}(x_1) \tag{5.19}$$

$$\frac{\partial}{\partial x^{\sigma}_2}f^{c,R}_{\rho}(x_1,x_2)-\frac{\partial}{\partial x^{\rho}_2}f^{c,R}_{\sigma}(x_1,x_2)=F_{\sigma\rho}(x_2) \tag{5.20}$$

but, in general, they are not identical:

$$f^{c,R}_{\mu}(x_1,x_2)\neq f^{c,L}_{\mu}(x_1,x_2) \tag{5.21}$$

This casts some doubt on the general applicability of Schwinger replacement [2]

$$\widetilde{G}(x_1,x_2)\longrightarrow < x_1 \mid \widetilde{G} \mid x_2 > \tag{5.22}$$

with $\widetilde{G}$ treated as the same operator in both eqns. (5.15) and (5.16).

6 Ponderomotive gauges in the canonical formalism - Maxwell theory

We consider now the use of our ponderomotive gauges in the quantized field theoretic context i.e. in QED. We shall show that our gauge constraint and Gauss's law are canonically conjugate leading to a rather simple form for the Dirac brackets of the theory. The canonical Hamiltonian in Maxwell theory

$$H_c=\int d^3x\left[\frac{1}{2}\vec{\Pi}^2+\frac{1}{2}\vec{B}^2+A^0\,\vec{\nabla}\cdot\vec{\Pi}\right] \tag{6.1}$$

with the primary constraints:

$$D_1=\Pi^0=0 \tag{6.2}$$

$$-D_2=\partial_i\Pi_i=0 \tag{6.3}$$

requires two futher constraints before the construction of the Dirac brackets [11]
Let us choose them to be a temporal gauge condition

$$D_3 = A_0 = 0 \tag{6.4}$$

and ponderomotive space-like gauge condition

$$D_4 = \int_0^1 A_i(\vec{c}(\vec{x}, \tau)) \frac{\partial c_i}{\partial \tau} d\tau = 0 \tag{6.5}$$

at a given moment of time. The curve c runs from $\vec{c}(\vec{x}, 0) = \vec{x}_0$ to $\vec{c}(\vec{x}, 1) = \vec{x}$
The corresponding Hamiltonian

$$H = H_c + \int d^3x v_2 \partial_i \Pi_i \tag{6.6}$$

where

$$v_2 = \int_0^1 \Pi_i(c) \frac{\partial c_i(\vec{x}, \tau)}{\partial \tau} d\tau \tag{6.7}$$

weakly commutes with the constraints D_i.
The Poisson brackets between the D_j turn out to be:

$$c_{31} = -c_{13} \equiv [D_3(\vec{x}), D_1(\vec{y})]_P = \delta_3(\vec{x} - \vec{y}) \tag{6.8}$$

$$c_{42}(x, y) = -c_{24}(y, x) \equiv [D_4(\vec{x}), D_2(\vec{y})]_P = \delta_3(\vec{y} - \vec{x}) - \delta_3(\vec{y} - \vec{x}_0) \tag{6.9}$$

The rest of Poisson brackets vanish. As an example let us derive eqn.(6.9). We have

$$c_{42} = [D_4(x), D_2(y)]_P = -\frac{\partial}{\partial y_k} \int_0^1 d\tau \frac{\partial c_k}{\partial \tau} \delta(y_1 - c_1) \delta(y_2 - c_2) \delta(y_3 - c_3) \tag{6.10}$$

Hence

$$\begin{aligned} c_{42} &= \int_0^1 d\tau \frac{\partial c_i}{\partial \tau} \frac{\partial}{\partial c_i} \delta_3(\vec{y} - \vec{c}(\vec{x}, \tau)) d\tau \\ &= \delta_3(\vec{y} - \vec{c}(\vec{x}, 1)) - \delta_3(\vec{y} - \vec{c}(\vec{x}, 0)) \\ &= \delta_3(\vec{y} - \vec{x}) - \delta_3(\vec{y} - \vec{x}_0) \end{aligned} \tag{6.11}$$

If now we discard the single point $\vec{x} = \vec{y} = \vec{x}_0$ the c_{ik} can be written as

$$c_{ik} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \cdot \delta_3(\vec{x} - \vec{y})$$

so that

$$c_{ik}^{-1} = -c_{ik} \tag{6.12}$$

The simplicity of the c_{ik} stems from the choice of the ponderomotive gauge condition D_4; moreover, and this is very important, D_4 and D_2 (Gauss's law) are canonically conjugate.
The Dirac brackets of the theory,

$$[A,B]_D = [A,B]_P - [A,D_i]_P\ c_{ik}^{-1}\ [D_k,B]_P \tag{6.13}$$

become in our case

$$\begin{aligned} [A,B]_D = [A,B]_P + [A,D_3]_P \cdot [D_1,B]_P - \\ -[A,D_1]_P \cdot [D_3,B]_P + \\ +[A,D_4]_P \cdot [D_2,B]_P - [A,D_2]_P \cdot [D_4,B]_P \end{aligned} \tag{6.14}$$

For $\vec{\Pi}$, $\vec{A}$ the Poisson brackets with D_i's are:

$$[A_i(\vec{x}), D_2(\vec{z})]_P = \frac{\partial}{\partial x^i}\delta_3(\vec{x}-\vec{z}) \tag{6.15}$$

$$[\Pi_i(\vec{x}), D_4(\vec{z})]_P = -\int_0^1 \delta_3(\vec{x}-\vec{c}(\vec{z},\tau))\frac{\partial c^i}{\partial \tau}d\tau \tag{6.16}$$

and the other brackets vanish.
So, finally, for

$$\begin{aligned} \vec{x} \neq \vec{x}_0 \\ \vec{y} \neq \vec{y}_0 \end{aligned} \tag{6.17}$$

$$\begin{aligned} [\Pi_i(x), A_j(y)]_D = -\delta_{ij}\delta(\vec{x}-\vec{y}) + [\Pi_i, D_4]_P \cdot [D_2, A_j]_P \\ = -\delta_{ij}\delta(\vec{x}-\vec{y}) + \int d^3z \int d\tau \delta_3(\vec{x}-\vec{c}(\vec{z},\tau))\frac{\partial c^i}{\partial \tau}\partial_j\delta_3(\vec{y}-\vec{z}) \\ = -\delta_{ij}\delta(\vec{x}-\vec{y}) + \frac{\partial}{\partial y_j}\int d\tau\delta_3(\vec{x}-\vec{c}(\vec{y},\tau))\frac{\partial c^i}{\partial \tau}d\tau \end{aligned} \tag{6.18}$$

The other Dirac brackets vanish. Π^0 and A^0 being constraints, have vanishing Diracs brackets with any variable.

7 Dirac brackets for Y-M theory

In the case of Yang-Mills theory we limit ourselves to self-contractible families, defined by eqn.(3.11). They have a useful property established earlier (compare Lemma in the proof of Theorem 2), namely Y-M potentials are orthogonal to these curves, i.e. from the gauge constraints

$$\int_{c(x,x_0)} A^\mu_a(y) dy_\mu = 0 \tag{7.1}$$

follows eqn.(3.18)

$$\frac{\partial c_\mu(x, x_0, \tau)}{\partial \tau} A^\mu_a(c(x, x_0, \tau)) = 0 \tag{7.2}$$

for any $0 \leq \tau \leq 1$ and $x \in V$. This relation will allow us to establish eq.(7.14) which in turn is a crucial condition responsible for the simple form of Dirac brackets.

We are going to implement these gauges into canonical formalism of Y-M theory.

In what follows the discussion of surface terms will be omitted. The canonical Hamiltonian is then:

$$H = \int_V d^3x \mathcal{H} \tag{7.3}$$

with

$$\mathcal{H} = \frac{1}{2}(\vec{B}_a \cdot \vec{B}_a + \vec{E}_a \cdot \vec{E}_a) - [\vec{\nabla} \cdot \vec{E}_a - g\mathcal{C}_{abc}\vec{A}_b \cdot \vec{E}_c]A^0_a \tag{7.4}$$

$$E^i_a(\vec{x}) = -\Pi^i_a(\vec{x}) \tag{7.5}$$

$$D^{(1)}_a = \Pi^0_a \approx 0 \tag{7.6}$$

$$D^{(2)}_a = \vec{\nabla} \cdot \vec{E}_a - g\mathcal{C}_{abc}\vec{A}_b \cdot \vec{E}_c \approx 0 \tag{7.7}$$

We take temporal gauge

$$D^{(3)}_a = A^0_a \approx 0 \tag{7.8}$$

and ponderomotive space-like gauge constraint

$$D^{(4)}_a = \int_{c(\vec{x},\vec{x}_0)} A^i_a(y) dy^i \approx 0 \tag{7.9}$$

The constraints (7.6-7.9) are compatible with

$$\mathcal{H}' = \frac{1}{2}(\vec{B}_a \cdot \vec{B}_a + \vec{E}_a \cdot \vec{E}_a) + D_a^{(2)}(x) v_{(a)}^{(2)}(x) \tag{7.10}$$

where

$$v_a^{(2)}(x) = \int_{c(\vec{x},\vec{x}_0)} dy^i E_a^i(y) \tag{7.11}$$

Our further considerations will be valid for the region V_-:

$$V_- = V - P(\vec{x}_0) \tag{7.12}$$

Next, let us remark that compatibility of (7.9) with (7.10) is evident once we prove - in analogy with Maxwell theory - that in V_-

$$[D_d^{(4)}(\vec{x}), D_a^{(2)}(\vec{y})]_P = \delta(\vec{x} - \vec{y})\delta_{ad} \tag{7.13}$$

The first term of $D_a^{(2)}(y)$, $-\vec{\nabla} \cdot \vec{E}_a$ (comp. eq.(7.7))yields already r.h.s. of (7.13) - derivation is the same as for Maxwell theory. So we have to show that

$$\mathcal{C}_{abc}[D_a^{(4)}(\vec{x}), A_b^i E_c^i(y)]_P \approx 0 \tag{7.14}$$

The use of (7.9) gives

$$\begin{aligned} \mathcal{C}_{abc}[D_a^{(4)}(x), A_b^i(y) E_c^i(y)]_P = \\ = \mathcal{C}_{abc} \int_{c(\vec{x},\vec{x}_0)} dz^k [A_d^k(z), A_b^i(y) E_c^i(y)]_P = \\ = \mathcal{C}_{abc}(-)\delta_{cd} \int_{c(\vec{x},\vec{x}_0)} dz^k \delta(\vec{z} - \vec{y}) A_b^k(z) \end{aligned} \tag{7.15}$$

Please notice, that $dz^k A^k(z)|_{z \in c(\vec{x},\vec{x}_0)} \approx 0$ from (7.9) (comp.eqns (7.1), (7.2)), therefore (7.14) is proved.
Let us come back to constraints ((7.6)-(7.9)). With the help of (7.13)the matrix

$$d_{a,b}^{i,k} = [D_a^{(i)}(x), D_b^{(k)}(y)]_P \tag{7.16}$$

can be written as

$$d_{a,b}^{i,k} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}^{ik} \cdot \delta_{ab}\delta_3(\vec{x} - \vec{y}) \tag{7.17}$$

so that

$$d^{-1} = -d \tag{7.18}$$

and Dirac brackets of the theory follow [11]:

$$[E^r_c(\vec{x}), A^s_d(\vec{y})]_D = \delta_{cd}\delta_{rs}\delta(\vec{x}-\vec{y}) -$$
$$- \left[\frac{\partial}{\partial y^s}\delta_{cd} - g\mathcal{C}_{cdb}A^s_b(y)\right] \cdot \int_{w\in c(\vec{y},\vec{x}_0)} dw^r \delta(\vec{x}-\vec{w}) \tag{7.19}$$

$$[A^r_c, A^s_d]_D = 0 \tag{7.20}$$

$$[E^r_c(\vec{x}), E^s_d(\vec{y})]_D =$$
$$= g\mathcal{C}_{cdf}\left[\int_{w\in c(\vec{x},\vec{x}_0)} dw^s \delta(\vec{y}-\vec{w})E^r_f(\vec{x}) + \int_{w\in c(\vec{y},\vec{x}_0)} dw^r \delta(\vec{x}-\vec{w})E^s_f(y)\right] \tag{7.21}$$

In the next section eqns.(7.19 - 7.21) will be used in the derivation of non-abelian algebras.

8 Flux algebras

In this chapter we are going to establish algebras of fluxes:

$$\mathcal{B}^{(\sigma)}_a = \int_\sigma B^k_a n^k d\sigma \tag{8.1}$$

$$\mathcal{E}^{(\sigma^*)}_a = \int_{\sigma^*} E^k_a n^k d\sigma \tag{8.2}$$
$$\vec{x}^0 \notin \sigma$$

The surfaces σ, σ^* are not interrelated. In the first part of this chapter we shall choose these surfaces in such a manner that the fluxes $\mathcal{B}$, $\mathcal{E}$ will be equal to loop integrals over the potential and dual potential, respectively.
Let us consider at the beginning a special type of surfaces appearing in definitions (8.1), (8.2) of $\mathcal{B}$, $\mathcal{E}$ fluxes. Take a loop L and some homotopy $c(\vec{x},\vec{a})$. We define a horn $H(L,c)$:

$$\vec{x} \in H(L,c) \Longleftrightarrow x^k = c^k(\vec{L}(t),\vec{a},t_1) \tag{8.3}$$

for some $t, t_1 \in [0,1]$ and fix orientation on this surface:

$$\vec{n}|| \left(\frac{\partial \vec{c}}{\partial t_1} \times \frac{\partial \vec{c}}{\partial t} \right) \tag{8.4}$$

We are going to show that fluxes $\mathcal{B}, \mathcal{E}$ through these homotopy horns are equal to loop integrals:

$$\int_{H(L,c)} (\vec{B}_a \cdot \vec{n}) d\sigma = \int_L f_a^r dx^r \tag{8.5}$$

$$\int_{H(L^*,c^*)} (\vec{E}_a \cdot \vec{n}) d\sigma = \int_L .^* f_a^r dx^r \tag{8.6}$$

with

$$f_a^r(x) = \int_c B_a^k \varepsilon^{kij} \frac{\partial y^j}{\partial x^r} dy^i \tag{8.7}$$

$$^* f_a^r(x) = \int_{c^*} E_a^k \varepsilon^{kij} \frac{\partial y^j}{\partial x^r} dy^i \tag{8.8}$$

L, L^* and c, c^* need not be related. At this stage we need not specify in what gauge $\vec{B}_a, \vec{E}_a$ are given. Eqns.(8.5), (8.6) are consequence of a simple observation. Take any antisymmetric tensor T_{ij} and define

$$g^r(x) = \int_{y \in c(\vec{x}, \vec{a})} T^{ij}(y) \frac{\partial y^j}{\partial x^r} dy^i \tag{8.9}$$

Then

$$\int_L g^r(x) dx^r = \int dt dt_1 \frac{\partial y^i}{\partial t_1} \frac{\partial y^j}{\partial t} \varepsilon^{ijk} T^k(y) \tag{8.10}$$

where

$$T^{ij} = \varepsilon^{ijk} T^k \tag{8.11}$$

$$y^i = c^i(L(t), x_0, t_1) \tag{8.12}$$

$$\int_L g^r(x) dx^r = \int_{H(L,c)} (\vec{T} \cdot \vec{n}) d\sigma \tag{8.13}$$

Replacement $g \longrightarrow f or^* f$ and $T^k \longrightarrow B^k or E^k$, gives eqns.(8.5) and (8.6), respectively.

If we specify $\vec{B}_a$, $\vec{E}_a$ to be in a gauge defined through c from eqn.(8.7), then

$f_a^r(x)$ is a potential in this gauge. Still there is a vast choice of homotopies c^* defining dual potential $,^* f$. Let us consider Dirac brackets of $\mathcal{E}, B$ in c-gauge:

$$
\begin{aligned}
[\mathcal{B}_d, \mathcal{E}_c]_D &= \left[\int_L f_d^r dx^r, \int_{L^*} .^* f_c^s dy^s\right]_D = \\
&= \left[\int_L f_d^r dx^r, \int_{L^*} dy^s \int_{c^*(y,a^*)} dz^i \frac{\partial z^j}{\partial y^s} \varepsilon^{ijk} E_c^k(z)\right]_D
\end{aligned}
\tag{8.14}
$$

Using (7.19) one gets, after some algebra, the following expression:

$$
\begin{aligned}
\left[\mathcal{E}_c^{H^*}, \mathcal{B}_d^H\right]_D &= \delta_{cd} N(L; H^*) + \\
+g c_{cdg} &\int_{x \in L} dx^r f_g^r(x) N(c(x,a); H^*)
\end{aligned}
\tag{8.15}
$$

where

$$
N(L; H^*) = \sum_t sgn\left(\frac{\partial \vec{L}(t_1)}{\partial t_1} \cdot \vec{n}_{H^*}(t_2, t_3)\right) \tag{8.16}
$$

$$
N(c(\vec{x}, a); H^*) = \sum_{\tau(x)} sgn\left(\frac{\partial \vec{c}(\vec{x}, \vec{a}, \tau_1)}{\partial \tau_1} \cdot \vec{n}_{H^*}(\tau_2, \tau_3)\right) \tag{8.17}
$$

with t_i, $\tau_i(x)$ being the solutions of the following equations:

$$
\vec{c}^*(L^*(t_2), \vec{a}^*, t_3) = \vec{L}(t_1) \tag{8.18}
$$

$$
\vec{c}^*(L^*(\tau_2), \vec{a}^*, \tau_3) = \vec{c}(\vec{x}, \vec{a}, \tau_1) \tag{8.19}
$$

and $\vec{n}_{H^*}$ being normal to a horn $H^* \equiv H(L^*, c^*)$ (comp.eqns (8.3), (8.4)). The conditions (8.18) or (8.19) are fulfilled whenever the surface of H^* is pierced by loop L or homotopy curve $c(\vec{x}, \vec{a})$, respectively. N's in eqns (8.16), (8.17) denote net numbers of piercings.

The abelian part of (8.15) has been already discussed [7] for the radial gauge; for abelian theories it leads to t'Hooft algebra [12]. The non-abelian part can be expressed through surface integrals. Call K_N part of a loop L, characterized by $N(c; H^*) = N$, N fixed (K_N can consist of disjoint pieces). We have $L = \sum_N K_N$ and corresponding horn surface:

$$
H(L, c) = \bigcup_N H(K_N, c) \tag{8.20}
$$

where

$$x \in H(K_N, c) \Longleftrightarrow \vec{x} = \vec{c}(\vec{y}, \vec{a}, t) \tag{8.21}$$

for some $Y \in K_N$ and $t \in [0,1]$. Evidently eqn(8.5) holds for $H(K_N, c)$ so that eqn(8.15) can be rewritten as:

$$\begin{aligned}[\mathcal{E}_c^{H^*}, \mathcal{B}_d^H] = \delta_{cd} N(L; H^*) + \\ + g c_{cdg} \sum_N N \int_{H(K_N,c)} \vec{B}_g \cdot \vec{n} d\sigma \end{aligned} \tag{8.22}$$

Let us add, that in fact eqn(8.22) holds for any surface S, not necessarily a horn $H^*(L^*, c^*)$. H^* is useful if we want to keep relation with loop integrals over dual potential (see eqns (8.6), (8.8)). More generally, we have:

$$[\mathcal{E}_c^S, \mathcal{B}_d^H] = \delta_{cd} N(L; S) + g c_{cdf} \sum_N N \int_{H(K_N,c)} \vec{\mathcal{B}}_f \cdot \vec{n} d\sigma \tag{8.23}$$

Let us consider now fluxes $\mathcal{E}_c^{(S_1)}$, $\mathcal{E}_c^{(S_2)}$. Surfaces S_i are parametrized by given $s_i(t_1, t_2)$:

$$x \in S_i \Longleftrightarrow \vec{x} = \vec{s}_i(t_1, t_2) \tag{8.24}$$

for some $(t_1, t_2) \in [0,1]$.

The Dirac bracket of $\mathcal{E}_c^{(S_1)}$, $\mathcal{E}_d^{(S_2)}$ - calculated in c-gauge - is given by the following expression:

$$\begin{aligned}[\mathcal{E}_c^{(S_1)}, \mathcal{E}_d^{(S_2)}] = \\ g C_{cdf} \Big[\int_{s_1 \in S_1} \vec{E}_f(s_1) \vec{n}_{S_1}(s_1) N(c(\vec{s}_1, \vec{a}); S_2) d\sigma + \\ + \int_{s_2 \in S_2} \vec{E}_f(s_2) \vec{n}_{S_2}(s_2) N(c(\vec{s}_2, \vec{a}); S_1) d\sigma \Big] \end{aligned} \tag{8.25}$$

where N's are the net numbers of piercings:

$$N(c(\vec{s}_1, \vec{a}); S_2) = \sum_{t_i(s_1)} sgn \left(\frac{\partial \vec{c}(\vec{s}_1, \vec{a}, t_3)}{\partial t_3} \cdot \vec{n}_{S_2}(t_1, t_2) \right) \tag{8.26}$$

with $t_i(s_1)$ being solutions of the following equation:

$$\vec{c}(\vec{s}_1, \vec{a}, t_3) = \vec{s}_2(t_1, t_2) \tag{8.27}$$

Eqn (8.27) is fulfilled whenever, for a given $s_1 \in S_1$ the homotopy curve $c(\vec{s}_1, \vec{a})$ crosses the surface S_2. Changing $s_1 \to s_2$, $S_1 \to S_2$ in eqns (8.26), (8.27) one gets N from the second integral on the r.h.s. of eqn(8.25). In what follows we shall limit ourselves to surfaces being tangent at most along a curve to any convolution of homotopy curves $c(\vec{x}, \vec{a})$ considered in a given gauge. This in turn means that the net number of piercings given by eqn.(8.26) is undefined at most along a curve so that surface integrals in eqn.(8.25) may have a meaning. Making in (8.25) transition $S_2 \to S_1$ we get for $S_1 = S_2 = S$:

$$[\mathcal{E}_c^{(S)}, \mathcal{E}_d^{(S)}]_D = 2gc_{cdf} \int_{s\in S} \vec{E}_f(s) \cdot \vec{n}_S(s) N(c(s,a);S) d\sigma \tag{8.28}$$

In this case there is always at least one common point of $c(\vec{s}, \vec{a})$ and S, as $c(\vec{s}, \vec{a})$ ends on $s \in S$. The weight of this end- point contribution to N is $\frac{1}{2}$ as can be seen from the limiting transition $S_1 \to S_2$ in eqn(8.25). Therefore, for any fixed $s \in S$:

$$\begin{aligned} 2N(c(s,a);S) &= sgn \frac{\partial \vec{c}(\vec{s}, \vec{a}, t_3)}{\partial t_3}|_{t_3=1} \cdot \vec{n}_S(\vec{s}) + \\ &+2 \sum_{t;t_3\neq 1, s\neq s(t_1,t_2)} sgn \left(\frac{\partial c(\vec{s}, \vec{a}, t_3)}{\partial t_3} \cdot \vec{n}_S(t_1,t_2) \right) \end{aligned} \tag{8.29}$$

with

$$\vec{c}(\vec{s}, \vec{a}, t_3) = \vec{s}(t_1, t_2) \tag{8.30}$$

Eqns (8.23), (8.28) together with the trivial bracket

$$[\mathcal{B}_c, \mathcal{B}_d]_D = 0 \tag{8.31}$$

do not form closed algebra for any chosen $H(L,c)$ and S. They can be however replaced by a set of closed algebras on the properly chosen parts of $H(L,c)$ and S. This will not be discussed here. Let us conclude with a choice of such $H(L_0,c)$ and S_0 that

$$L_0 \bigcup S_0 = \emptyset \tag{8.32}$$

i.e. abelian part does not contribute to (8.23). Moreover, put $2N$ in eqn(8.28) and N in eqn(8.23) equal to 1. (Example: in the Fock-Schwinger gauge take

$H(L_0, C)$ to be a cone and S_0 to be any planar surface containing elliptic section of H_0). In such a case we have:

$$[\mathcal{E}_a^{S_0}, \mathcal{B}_b^{H_0}]_D = gc_{abc}\mathcal{B}_c^{H_0} \tag{8.33}$$

$$[\mathcal{E}_a^{S_0}, \mathcal{E}_b^{S_0}]_D = gc_{abc}\mathcal{E}_c^{S_0} \tag{8.34}$$

$$[\mathcal{B}_a^{H_0}, \mathcal{B}_b^{H_0}]_D = 0 \tag{8.35}$$

If we took S^0 to be closed surface surrounding $\vec{a}$, then (8.34) still holds and is the algebra of charges contained in its interior, V_0. The question whether the algebra holds for any closed surface will be discussed in the next chapter.

9 Local charge algebras

It seems interesting to investigate whether colour electric flux algebra from (8.34) can be extended to the general case of two arbitrary closed surfaces; if it were so, then we might be able to establish local colour charge algebras smilar to well known flavour current algebras of the traditional quark model. Let us take closed surfaces $S_i \in V_-$ ($V_- = V - P(\vec{a})$, comp. eqn.(7.12)) which are boundaries of 3-dimensional, open regions $V_i \in V$. Next, denoting an outward flux of $\vec{E}_a$ through S_i as $\mathcal{E}_a(V_i)$, we shall check commutation relations of colour electric fluxes

$$\mathcal{E}_a(V_i) =_{def} \mathcal{E}_a^{S_i} \tag{9.1}$$

for two situations: a)$V_1 = V_2$, b) $V_1 \cap V_2 = \emptyset$.
For the case a) we get from eqns.(8.28-8.30)

$$\text{a)} \quad V_1 = V_2$$

$$if \quad P(\vec{a}) \in V_1, \quad [\mathcal{E}_a(V_1), \mathcal{E}_b(V_1)]_D = gc_{abc}\mathcal{E}_c(V_1) \tag{9.2}$$

$$if \quad P(\vec{a}) \notin V_1, \quad [\mathcal{E}_a(V_1), \mathcal{E}_b(V_1)]_D = -gc_{abc}\mathcal{E}_c(V_1) \tag{9.3}$$

For the case b) we get from eqns.(8.25-8.27)

$$\text{b)} \quad V_1 \cap V_2 = \emptyset$$

$$if \quad P(\vec{a}) \in V_1, P(\vec{a}) \notin V_2, \quad [\mathcal{E}_a(V_1), \mathcal{E}_b(V_2)]_D = g c_{abc} \mathcal{E}_c(V_2) \tag{9.4}$$

$$if \quad P(\vec{a}) \notin V_1 V_2, \quad [\mathcal{E}_a(V_1), \mathcal{E}_b(V_2)]_D = 0 \tag{9.5}$$

If we assumed that $\vec{E}$ is regular at $\vec{r} = \vec{a}$, then eqn.(9.4) would mean that the electric charges defined by (9.1) do not form local algebra. However, such an assumption is not sound; eqns.(9.2-9.5) exhibit essential role of the point $P(\vec{a})$ despite the fact that $P(\vec{a}) \notin S_i$ in our derivations. Having this in mind let us consider a sequence $\{V_0^k\}$ of open balls containing $\vec{a}$. Let their boundaries S_0^k have radii $\varepsilon_k \longrightarrow 0$ for $k \longrightarrow \infty$ (please notice that $\varepsilon_k = 0$ is never reached as $S_0^k \subset V_-$). Let us define

$$e_c^k = \int_{S_0^k} \vec{E}_c \vec{n} d\sigma \tag{9.6}$$

and assume existence of limit

$$e_c = \lim_{k \longrightarrow \infty} e_c^k \tag{9.7}$$

For any fixed surface S eqns.(8.25-8.27) lead to the following relation

$$[e_a, \mathcal{E}_b^S]_D =_{def} \lim_{k \longrightarrow \infty} [e_a^k, \mathcal{E}_b^S]_D = g c_{abc} \mathcal{E}_b^S \tag{9.8}$$

and, as a consequence,

$$[e_a, \mathcal{E}_b(V_i)]_D = g c_{abc} \mathcal{E}_c(V_i) \tag{9.9}$$

Moreover,

$$[e_a, e_b]_D =_{def} \lim_{k,r \longrightarrow \infty} [e_a^k, e_b^r]_D = g c_{abc} e_c \tag{9.10}$$

and, due to eqn.(7.19),(7.21)

$$[e_a, A_b^i(\vec{x})]_D = g c_{abc} A_c^i(\vec{x}) \tag{9.11}$$

$$[e_a, E_b^i(\vec{x})]_D = g c_{abc} E_c^i(\vec{x}) \tag{9.12}$$

for any $\vec{x} \in V_-$.

Now we are in a position to introduce charges associated with the new field T_a defined below

$$Q_a^T(V_1) = \int_{S_1} \vec{T}_a \cdot \vec{n} d\sigma \tag{9.13}$$

$$\vec{T}_a(\vec{r}) = \vec{E}_a(\vec{r}) - \frac{1}{4\pi} \frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3} e_a, \quad \vec{r} \in V_- \tag{9.14}$$

so that

$$Q_a^T(V_1) = \mathcal{E}_a(V_i) - g(V_1)e_a \tag{9.15}$$

where

$$\begin{aligned} g(V_1) &= 1 \quad if\ \vec{a} \in V_1 \\ g(V_1) &= 0 \quad if\ \vec{a} \notin V_1 \end{aligned} \tag{9.16}$$

Using eqns.(9.1-9.16) one can check that $Q_a^T(V_i)$ satisfy local algebra

$$[Q_a^T(V_1), Q_b^T(V_1)]_D = -g c_{abc} Q_c^T(V_1) \tag{9.17}$$

$$\text{for any fixed open } V_1 \subset V$$

and

$$[Q_a^T(V_1), Q_b^T(V_2)]_D = 0 \tag{9.18}$$

$$\text{for } V_1 \cap V_2 = 0$$

Eqns.(9.2), (9.3) together with eqns.(9.14-9.18) clearly demonstrate existence of a point term at $\vec{r} = \vec{a}$. This leads to non-integrable singularity in Hamiltonian, unless $e_a = 0$. In quantum theory it would mean that we choose such a subspace of states, $\mathcal{H}$, that

$$e_a|\Psi >= 0 \tag{9.19}$$

for any $|\Psi > \in \mathcal{H}$ and for any "a".
Then eqns.(9.11),(9.12) lead to

$$E_a^i(\vec{x})|\Psi >= A_a^i(\vec{x})|\Psi >= 0 \tag{9.20}$$

for any $\vec{x} \in V_-$ and any $|\Psi > \in \mathcal{H}$, i.e. theory is trivial unless $e_a \neq 0$. Therefore in order to avoid obstruction to the Gauss law we have to exclude a curve from the region V and to deal with the region V_{-c_0}

$$V_{-c_0} = V - \{c(\vec{R}_0 \in S, \vec{a})\} \tag{9.21}$$

where $c_0 = c(\vec{R}_0 \in S, \vec{a})$ is any curve joining $\vec{a}$ with some chosen point $\vec{R}_0$ at the boundary S of the region V. The curve c_0 should belong to a family of curves used for gauge fixing so that nonabelian parts of Dirac brackets (eqns.(7.19),(7.21)) would not mix dynamical variables on c_0 with those on

V_{-c_0}. The choice of V_{-c_0} makes the use of e_a unnecessary as closed surfaces around $\vec{a}$ cannot be contained in V_{-c_0}. Now, whether curve c_0 will be a regular or singular line of field E can be in principle derived from dynamics of the theory in the region V_{-c_0}.

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A Appendix

We shall show that $A = f^c_\rho$ given by eqn.(3.17) of theorem 2 satisfies eqn.(3.1) if eqn.(3.4) is satisfied.
Let us start with formula

$$F_{\alpha\rho} = f^c_{\rho,\alpha} - f^c_{\alpha,\rho} = G_{\alpha\rho} - \int \frac{\partial c^\mu}{\partial x^\rho}\frac{\partial c^\sigma}{\partial x^\alpha}\frac{\partial c^\nu}{\partial t} \cdot [G_{\mu\nu,\sigma} + G_{\nu\sigma,\mu} + G_{\sigma\mu,\nu}]dt \quad \text{(A.1)}$$

which is obtained from (3.17) in a manner similar to that applied in the proof of theorem 1.
Next, using (3.4) we get from (A.1):

$$G_{\alpha\rho} - F_{\alpha\rho} = -ig\left[[f^c_\sigma, G_{\mu\nu}] + [f^c_\mu, G_{\nu\sigma}]\right] \quad \text{(A.2)}$$

(The term $[f^c_\nu, G_{\sigma\mu}]$ does not contribute, because $\frac{\partial c}{\partial t} \cdot f^c = 0$).
Substituting f^c from eqn.(3.17) we get

$$G_{\alpha\rho} - F_{\alpha\rho} = -ig(I_{\rho\alpha} - I_{\alpha\rho}) \quad \text{(A.3)}$$

where

$$I_{\rho\alpha} = \int \frac{\partial c^\mu_t}{\partial x^\rho}\frac{\partial c^\sigma_t}{\partial x^\alpha}\frac{\partial c^\nu_t}{\partial t}\frac{\partial c^\delta(c_t, x_0, \tau)}{\partial c^\sigma_t}\frac{\partial c^\beta(c_t, x_0, \tau)}{\partial \tau} dt d\tau [G_{\beta\delta}(c(c_t, x_0, \tau)), G_{\mu\nu}(c_t)] \quad \text{(A.4)}$$

and $c_t = c(x, x_0, t)$
Applying eqn.(3.15) we get

$$I_{\rho\alpha} = \int_0^1 dt \int_0^t dh \frac{\partial c_t^\mu}{\partial x^\rho} \frac{\partial c_t^\nu}{\partial t} \frac{\partial c_h^\delta}{\partial x^\alpha} \frac{\partial c_h^\beta}{\partial h} [G_{\beta\delta}(c_h), G_{\mu\nu}(c_t)] \tag{A.5}$$

and, after a suitable change of variables,

$$I_{\alpha\rho} = -\int_0^1 dh \int_0^h dt \frac{\partial c_t^\mu}{\partial x^\rho} \frac{\partial c_t^\nu}{\partial t} \frac{\partial c_h^\delta}{\partial x^\alpha} \frac{\partial c_h^\beta}{\partial h} [G_{\beta\delta}(c_h), G_{\mu\nu}(c_t)] \tag{A.6}$$

so that

$$I_{\rho\alpha} - I_{\alpha\rho} = [f_\rho^c, f_\alpha^c] \tag{A.7}$$

Then from (A.3)

$$G_{\alpha\rho} = F_{\alpha\rho} + ig[f_\alpha^c, f_\rho^c] \tag{A.8}$$

i.e. eqn.(3.1) as desired. Eqn.(3.6) trivially follows from (3.16); it is enough to repleace T by G in (3.17) and use the result (3.18) of the lemma.

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